Gain Scheduled LPV Systems - Global Vision and Stability Results

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***Abstract*— In this paper the different phases of Gain Scheduling are visited, with special attention to stability results for LPV systems. Given that the main reviews in the area date from 2000, more recent results were gathered. Stability is to be shown mainly with Parameter Dependent Lyapunov Functions, and a discussion on performance analysis is presented. At the end, a procedure for the Synthesis and Analysis of a Gain Scheduled LPV controller is presented based on the different results presented. This procedure is simpler than the purely theoretical ones, and still more reliable than pure simulation, as it is often the case in industrial applications.**

I. INTRODUCTION

A. Preliminaries

Gain Scheduling is one of the most popular approaches to non-linear and time-varying systems. It is particularly appreciated in industrial applications, mainly because of possible intuitive approaches and also for its simplicity of implementation. However, Gain Scheduling applications usually lack of stability theoretical results, and most of the time, for simplicity, simulations are used to prove system's stability and performance. This is the main motivation for this work, and this is why stability results for Gain Scheduled LPV Systems were gathered together.

Given that after the main review in the area, by Rugh and Shamma [11], new techniques were proposed we thought that a new review could be useful, and even though a thorough review is not our main goal in this present work, we will visit the different phases of Gain Scheduling for LPV Systems, giving special attention for what was done after 2000, mainly for stability results. Finally, a methodology for Gain Scheduling is proposed based on the concepts found in the literature, and it is, in our opinion, more reliable than simulations, as often used in industrial applications, and still simpler than the specific synthesis techniques.

A different approach of Gain Scheduling, with the velocity-based linearization, is given by [8].

B. Motivation

Our current research is in the control of tension and frequency of embedded electrical networks. In this particular type of network, each load that is connected to the system may have and important influence over tension and frequency, which is not the case in large power systems, where these parameters are imposed by the network.

A major problem of these type of networks lies on the fact that the system dynamic model changes with load variation (the loads are part of the model). In an embedded electrical network, the power needed by the loads can vary within a very large range according to the environment and use of the system. For example, in a ship, stabilisators will demand power that varies according to the speed and sea state, lights will be more used during the night, restaurants work in well-defined periods, etc.

Given that the major electrical loads that demand power (electric machines in a ship for example) are non-linear, we have a tricky system to control: a dynamic evolving non-linear model. Three approaches are possible:

- A Robust LTI controller, where the variations would be treated as uncertainties
- A Non-Linear technique
- Gain Scheduling

The first (LTI Robust Controller), depending on the system, would have to be so robust that its performances would be degradated. The second (Non-Linear technique) would certainly be useful, but it can be complicated and given the dynamic model time variation, it is not always suitable. The third one will then be used.

In this review, Gain Scheduling *methodologies* will be explored, on theirs general form.

II. BASES

A. Gain Scheduling

"Machines that walk, swim, or fly are Gain Scheduled" [11].

The general idea behind Gain Scheduling is very simple: a regulator that will evolve to match changes in the environment is gain scheduled. In a plane for example, speed and air pressure can be used as scheduling variables, and so the "optimum" regulator is chosen based in these parameters.

The main advantages of Gains Scheduling are [11]:

- Use of linear design tools for non-linear problems
- Few assumptions on the model (can even be used without an analytic model)
- Linear intuition can be used for the synthesis (in contrast to non-linear techniques that may use variable changes)
- Computation burden is usually inferior to non-linear design techniques.

On the other hand, one of the drawbacks is mainly the fact that there are not always stability and performance proofs (apart from simulations). Also, there is not a well defined methodology for Gain Scheduling in its general form, the solution may depend on the specific problem.

A classical approach to Gain Scheduling is based on the "divide-and-conquer" strategy. First of all, it consists in having several local models (for operating points or linearisations of a non-linear system) and then proceed to the synthesis of local regulators. With these regulators, one must create a law that will link them according to the scheduling variables (Gain/Pole/Zero interpolation or simple commutation for example). This will create a Gain Scheduled *non-linear* regulator.

A more systematic approach can be given if at the beginning there is an LPV model. This will be the topic of our next paragraphs.

B. Linear Parameter Varying (LPV) Model

The LPV model is an useful representation of non-linear or time-evolving model systems. Once this model is obtained, the synthesis of a controller is straightforward (even though stability proofs are not). An LPV standard model has the following structure [12]:

$$\begin{pmatrix} \dot{x} \\ \hline z_p \\ y \end{pmatrix} = \left(\begin{array}{c|cc} A(\delta(t)) & B_p(\delta(t)) & B(\delta(t)) \\ \hline C_p(\delta(t)) & D_p(\delta(t)) & E(\delta(t)) \\ C(\delta(t)) & F(\delta(t)) & 0 \end{array}\right) \begin{pmatrix} x \\ \hline w_p \\ u \end{pmatrix} \tag{1}$$

The signals have the usual representation: u is the command, y the measured output and $w_p \to z_p$ a performance channel. This represents a family of systems with $\delta : [0, \infty) \to \mathbb{R}^l$ with $\forall t \geq 0$. Each one of this systems (for a fixed value of $\delta(t)$) is linear, hence the name *Linear Parameter Varying*. The hypothesis is that $\delta(t)$ is not know *à priori* but can be measured (or at least evaluated) in real time. If the parameter $\delta(t)$ depends on at least one of the states variables, it is called a *quasi-LPV* model. This can be used to "disguise" non-linearities as a parameter to create an LPV model.

As this is a crucial phase of the procedure, it should not be neglected (as it sometimes is). This problem is addressed in [2], where two approaches are proposed to obtain a closer relation between the non-linear system and the LPV Model.

The first one is the *Deviation Approach*: it's logical to think that all the trajectories of the real system will also be a trajectory of the LPV model. However, the inverse is not always true: the LPV model is *larger* than the original. The *Deviation Approach* will try to minimize this difference.

The second one is the *Sensibility Approach*, which has as main idea the reduction of the influence of the parameter $\delta(t)$ in the LPV model. The reader is encouraged to refer to [2] for further details.

III. Stability Analysis

In this section the stability problem of closed-loop systems will be addressed, and all the results will be presented as LMI problems. This phase can be the most important of the synthesis procedure, given that it is possible to use linear techniques over a linearized system, which does not assure, *à priori*, any stability. The main results for Gain Scheduled LPV systems are given by the Lyapunov theory. There are, however, other results that will also be explored in this section.

As a motivation for the stability study, the following example is given [11]:

Example 3.1: **Local Stability x Global Stability**

When parameters (or uncertainties) vary with time, the fact that all the locals systems are stable does not imply global stability. This is not intuitive, but it is sadly true. This example illustrates this situation, as seen in Fig. (1). The dynamics are given by the matrix:

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -(1+\sigma(t)/2) & -0.2 \end{pmatrix} x \tag{2}$$

For all $\sigma(t) \in [-1, 1]$ all the eigenvalues of the system are stable, which implies that all the local systems are stable. However, for $\sigma(t) = \cos 2t$ through simulation it is obvious that there is an unstable dynamic for non-zero initial conditions. A physical explanation is that the stiffness increases as it is contracting and decreases as it is expanding. This way, oscillations tend to grow (Fig. (2)).

A. Stability in the Sense of Lyapunov

The principal stability results for LPV systems are in the sense of Lyapunov. In this paper results will be presented gradually, having as the starting point the classical Lyapunov theorems and definitions. First of all, we state Lyapunov's Theorem [12]:

Theorem 3.1: **(Lyapunov Theorem)** For the differential equation $\dot{x}(t) = f(x(t), t)$, considering $x^* \in \mathcal{X}$ an equilibrium point in the interior of the set $\mathcal{S}$:

(a) If there exists a positive-definite, continuously differentiable function $V : \mathcal{S} \times \mathcal{T} \to \mathbb{R}$ with $V(x^*, t) = 0$ and V' negative semi-definite, then x^* is stable. If V is decrescent, then x^* is Uniformly Stable.

(b) If there exists a positive-definite decrescent and continuously differentiable function $V : \mathcal{S} \times \mathcal{T} \to \mathbb{R}$ with $V(x^*, t) = 0$ and V' negative-definite, then x^* is Uniformly Asymptotically Stable.

This result is very well known, and it is presented here as the basis of the following sub-sections.

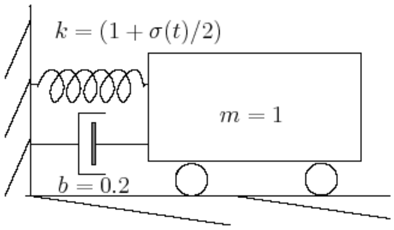

Fig. 1. Spring-Mass system with a time varying stiffness spring

1) Lyapunov Stability for LTI Systems:

Theorem 3.2: **Lyapunov Stability for LTI Systems** An LTI System is stable if and only if, there exists a symmetric positive definite matrix **P** ($P \in S^+$) such that, for the system $\dot{x} = Ax$:

$$A'P + PA < 0 \tag{3}$$

where the function $V(x) = x'Px$ is the *Lyapunov Function*

This condition is necessary and sufficient.

2) Lyapunov Stability for LPV Systems - Fixed Lyapunov Function: [1]

A similar theorem for LPV Systems can be stated, which is a natural extension of the latter.

Theorem 3.3: **Lyapunov Stability for LPV Systems** An LPV System is stable if there exists a symmetric positive definite matrix **P** ($P \in S^+$) such that, for the system $\dot{x} = A(\delta(t))x$:

$$A(\delta(t))'P + PA(\delta(t)) < 0 \tag{4}$$

where the function $V(x) = x'Px$ is the *Lyapunov Function*

This condition is now just sufficient [14], given that this is no longer the LTI case. No assumptions were made about the variation rate of the function, that is therefore unbounded. This is actually the case of *Switched Systems*, and a discussion on stability results for these systems is given in [10].

3) Lyapunov Stability for LPV Systems - Parameter-Dependent Lyapunov Function: If the derivative of the parameter is not considered for the search of a Lyapunov function, this search will concern a very large class of system, given that, in this case, this derivative can be unbounded. This conservatism may be responsible for the impossibility of finding a matrix **P** that verifies (4).

A less conservative approach is what is called a *Parameter Dependent Lyapunov Function*[2]. The idea is to have, rather than a matrix P, a matrix $P(\delta(t))$. In this case, Lyapunov's condition comes to [14]:

$$A'(\delta(t))P(\delta(t)) + P(\delta(t))A(\delta(t)) + \frac{dP}{dt} < 0 \tag{5}$$

which is equivalent to:

$$A'(\delta(t))P(\delta(t)) + P(\delta(t))A(\delta(t)) + \dot{\delta}\frac{dP}{d\delta} < 0 \tag{6}$$

This condition is once again just sufficient, but it is less conservative than the last. This is the main stability result for LPV systems using Lyapunov's Theory, and can be stated as follows [12]:

Theorem 3.4: **Parameter Dependent Lyapunov Theorem:** Assuming that the function $P : \Delta \to \mathbb{S}^p$ is continuously differentiable within a compact set Δ and that it verifies:

$$P(\delta) \succ 0 \qquad \forall \delta \in \Delta \tag{7}$$

$$\partial_\delta P(\delta)\lambda + A(\delta)^T P(\delta) + \tag{8}$$

$$+P(\delta)A(\delta) \prec 0 \quad \forall \delta \in \Delta \ and \ \lambda \in \Lambda \tag{9}$$

Then the origin of the system $\dot{x} = A(\delta(t))x(t)$ is exponentially stable for a variation of $\delta : \mathbb{R} \to \mathbb{R}^l$ that verifies for the subsets Δ and Λ of $\mathbb{R}^l$:

$$\delta(t) \in \Delta \quad et \ \lambda \in \Lambda \quad \forall t \in \mathbb{R} \tag{10}$$

In this case, the function $V(x, \delta) := x^T P(\delta)x$ is the *Quadratic Parameter Dependent Lyapunov Function* for the system.

In this theorem, λ is the maximum value of the derivative of the parameter. A problem of this theorem in particular, is that as the domains are continuous, there is an infinity of LMIs to evaluate. This is obviously impossible, so there are two possible solutions: make some structural assumption (such as polytopic domains, LFT structure, etc) or make a grid and test the conditions in all the chosen points. For the latter approach, the more points are used, the more reliable the result will be, but it is not an *analytical* result.

An interesting approach for analysis and synthesis of LPV systems with polynomial dependence is given in [16]. It is based in the Sum of Squares (SOS) decomposition. A polynomial $f(x)$ is *SOS* if there exists polynomial functions $f_1(x), ..., f_n(x)$ such that $f(x) = \sum_{i=1}^{n} f_i^2(x)$. This directly implies that $f(x) \geq 0$ for all $x \in \mathcal{R}^n$. The *SOS* decomposition assures a sufficient condition for non negativity of the multivariate polynomial and is equivalent to the existence of a positive semi-definite matrix Q and a properly chosen monomial vector $Z(x)$ such that $f(x) = Z'(x)QZ(x)$. This could help to provide a coherent methodology for the synthesis of a Lyapunov function for non-linear systems.

[1]For this work LPV systems are considered, but this proofs hold for uncertain systems as well.

[2]One of the first works in this sense is the PhD dissertation [14].

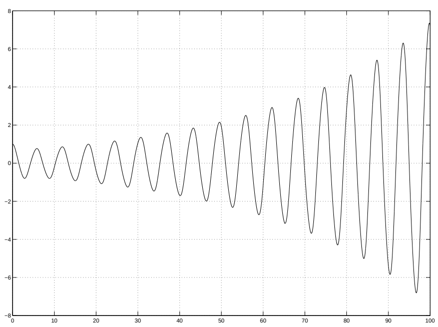

Fig. 2. Mass position as a function of time for $\sigma(t) = cos(2t)$ and non-zero initial conditions

B. Input/Output Stability: Small Gain Theory

Given that this work is focused on the Lypunov approach, this theory will not be explored. Nevertheless this is an interesting approach and specific LPV results are available in the literature. As it is an Input/Output approach, no assumptions are made on the model, that can be linear/non-linear, stationary/non-stationary, etc. The work [1] is a good starting point for further informations.

C. Multiple Models - Interpolated Systems

The works of Mohammed Chadli ([3], [4]) state stability results for *Multiple Models*, which are in fact a general form of interpolated systems. This is, from our point of view, very useful, because it can be an analytical proof for systems without a well-defined model (mainly for systems without an LPV model). Here the different steps of the procedure will be visited, and the main result will be stated.

First of all, considering the non-linear system:

$$\dot{x}(t) = f(x(t), u(t)) \tag{11}$$

For n arbitrary points $(x_i, u_i) \in \mathbb{R}^p \times \mathbb{R}^m$, the system can be linearized, and with a law that will be described in the following, a *multiple model* is defined as:

$$\dot{x}(t) = \sum_{i=1}^{n} \mu_i(z(t))(A_i x(t) + B_i u(t)) \tag{12}$$

where $\mu_i(z(t))$ are the *activation functions* and $z(t)$ is the decision variables vector, depending on the measurable state variables and eventually the command $u(t)$. The number of local models n depends on the desired accuracy, complexity of the non-linear system and on the structure of the activation functions. This is a general form of an interpolated system, because for each point there is a dependency on all the other local models, and not only on the ones close to the point.

Two hypothesis are stated:

Hypothesis 3.1: The activation functions $\mu_i(z(t))$ are continuously differentiable.

Hypothesis 3.2: The activation functions have the following properties:

$$\mu_i(z(t)) \geq 0 \tag{13}$$

$$\sum_{i=1}^{n} \mu_i(z(t)) = 1 \tag{14}$$

Stability is shown using *polyquadratic* Lyapunov functions depending only on the system's states. Consider the function:

$$V(x(t)) = x(t)^T P(x(t)) x(t) \tag{15}$$

with

$$P(x(t)) = \sum_{i=1}^{n} \mu_i(x(t)) P_i, P_i > 0 \tag{16}$$

The time derivative of the function (15) leads to

$$\begin{aligned} \dot{V}(x(t)) = \dot{x}(t)^T P(x(t)) x(t) + \\ + x(t)^T P(x(t)) \dot{x}(t) + x(t)^T \dot{P}(x(t)) x(t) \end{aligned} \tag{17}$$

There are the two "classical" Lyapunov terms and another one that depends on the derivative of P. This last one can be bounded as follows:

$$\begin{aligned} x(t)^T \dot{P}(x(t)) x(t) &= x(t)^T \sum_{i=1}^{n} \left\langle \frac{\partial \mu_i(x(t))}{\partial x(t)}, \frac{\partial x(t)}{\partial t} \right\rangle P_i x(t) \\ &\leq x(t)^T \sum_{i=1}^{n} \left| \left(\frac{\partial \mu_i(x(t))}{\partial x(t)} \right) \dot{x}(t) \right| P_i x(t) \end{aligned} \tag{18}$$

If it is possible to bound the term $\left| \left(\frac{\partial \mu_i(x(t))}{\partial x(t)} \right) \dot{x}(t) \right|$ without any dependence on the state (or its derivative):

Hypothesis 3.3: There exists a scalar $\upsilon > 0$ such that $\left| \left(\frac{\partial \mu_i(x(t))}{\partial x(t)} \right) \dot{x}(t) \right| \leq \upsilon, \forall x(t) \in \mathbb{R}^p, i \in \mathbb{I}_n$.

With these results it is finaly possible to state the theorem concerning stability of multiple models. From our point of view, even though there are several inequalities to verify, it is impressive to have an analytical proof for an interpolated system, and this is why we mention it in this present review.

Theorem 3.5: If there exists positive definite symmetric matrix Q and $P_i, i \in \mathbb{I}_n$ and symmetric matrix U and V such the following LMIs are verified:

$$P_i > P_{j+r}, i \in \mathbb{I}_n, j \in \mathbb{I}_{n-r} \tag{19}$$

$$A_i^T P_i + P_i A_i \leq U, \forall i \in \mathbb{I}_n \tag{20}$$

$$\begin{aligned} A_i^T P_j + P_j A_i + A_j^T P_i + P_i A_j \leq 2V, \\ \forall (i,j) \in \mathbb{I}_n^2, i < j \end{aligned} \tag{21}$$

$$U - V \leq 0 \tag{22}$$

$$V + r^{-1}(U - V) + \upsilon \sum_{i=1}^{r} P_i < -Q \tag{23}$$

with $\mu_i(z(t))\mu_j(z(t)) \neq 0$, r the maximal number of local models activated and υ an upward bound for the state variation (hypothesis (3.3)). Then, the equilibrium point of the multiple model (12) is globally exponentially stable.

Further details on stability results for discrete multiple models and applications are given in [4].

D. Contraction Analysis and δ stabilities

A last approach worth of mention is the *Contraction Analysis*. A review is given by [9], and several works were done by Jérôme Jouffroy, the base of this section being [7]. Here just the main idea of the theory will be stated. According to the author, it is a more general approach than Lyapunov's theory. The stability is shown not with equilibrium points, but *equilibrium trajectories*.

The main idea is to consider two neighbors trajectories initially separated by an infinitesimal distance. If they converge one toward the other, it is possible to imply the convergence of two trajectories separated by a finite distance. This is the base of the contraction analysis, to show the convergence toward an equilibrium trajectory. Later, with a δ stability (also called *incremental stability*) technique, it is possible to show the stability of the system.

IV. Performance Analysis

Once stability is verified, it is natural to search for the performances properties of the LPV system. Here the classical results based on the H_2 and H_∞ norms will be explored, as well as an unified approach for stability and performance.

It is important to keep in mind that local results for performance are simpler, but not always accurate for the global system (as seen in example (3.1) for stability). However, local results are indeed a good *indication* of the global performance, and it is intuitive to say that the better are the local performances, the better will the global be, even though this is not necessarily true. In the next section the extensions of these norms for LPV Systems will be explored.

A. H_2 and H_∞ Norms

An interesting method for performance evaluation is the norm between the input and output signals. It is possible to measure performance aspects such as trajectory tracking, noise rejection, control saturation, etc.

1) H_∞ norm extension for LPV systems: Let [11]:

$$z = T_{zv}^{\delta(.)} v \tag{24}$$

be the input/output relation of a closed-loop for an LPV system with zero initial conditions. The signal v is an exogenous signal and z is the error measure. The induced norm is then defined as [11]:

$$\|T_{zv}\|_{i,2} = \sup_{\delta(.) admissible} \sup_{v(.)} \frac{\left(\int_0^\infty z^T(t)z(t)dt\right)^{\frac{1}{2}}}{\left(\int_0^\infty v^T(t)v(t)dt\right)^{\frac{1}{2}}} \tag{25}$$

This is related to the "worst case scenario". It is not the usual system performance, but an inferior bound for it, as it is shown by the sup of all admissible trajectories of the sup of all admissible values (all the trajectories that verify (10)).

2) H_2 norm extension for LPV systems: The definition of the H_2 norm here developed [1] is the largest amplification caused by the *impulse response* for δ that verifies (10). Consider the m input LPV System:

$$S : \begin{cases} \dot{x} = A(\delta)x + [b_1(\delta), \dots, b_m(\delta)]w \\ z = C(\delta)x \end{cases} \tag{26}$$

It is possible to decompose this system in m subsystems, S_i where each input will be replaced for a non-zero initial condition that is equivalent to an impulse input.

$$S_i : \begin{cases} \dot{x_i} = A(\delta)x_i, \ x_i(0) = b_i(\theta_0) \\ z_i = C(\delta)x_i \end{cases} \tag{27}$$

The extension of the H_2 norm is then:

$$\gamma H_2(S) = \sup_{\delta(t)\in\Delta} \sum_{i=1}^{m} \int_0^\infty z_i^T(t)z_i(t)dt \tag{28}$$

As this concerns the output energy, it measures an "usual" performance of the system, the one that will be presented most of the time. These two last definitions generalize to LPV systems the two principal results for performance evaluation.

B. Stability and Performance Tests

Usually stability and performance are treated in different analysis. This is indeed an interesting approach, given that it is possible to make a rigorous stability analysis and a quicker local analysis for performance, for example. However, it is of course possible to evaluate both in a single inequality. This is done in several of our references, and an interesting approach is given to the problem of regulators interpolation that conserve the H_∞ performance in [17].

Consider a closed-loop LPV system:

$$\begin{cases} \dot{x}(t) = A(\delta(t))x(t) + B(\delta(t))w(t) \\ z(t) = C(\delta(t))x(t) \end{cases} \tag{29}$$

A sufficient condition for this closed-loop to have a H_∞ performance of γ and to be stable in the sense of Lyapunov is given by [17]:

Definition 4.1: For the closed-loop system (29), with $\delta \in \Delta \subset \mathbb{R}^l$, if there exists a symmetric positive semi-definite continuous matrix function $P(\delta)$ that verifies:

$$\begin{aligned} A^T(\delta)P(\delta) + P(\delta)A(\delta) + \frac{d}{dt}P(\delta) + \\ +\gamma^{-2}C^T(\delta)C(\delta) + P(\delta)B(\delta)B^T(\delta)P(\delta) < 0 \end{aligned} \tag{30}$$

then the closed looped system (29) has the H_∞ performance γ.

In our opinion, this is a very "reassuring" condition: it assures stability and a minimum performance level. If this condition is verified, it is possible to simulate the system to justify its performance in particular situations, because there is the certainty that even in unexpected situations, the worst performance possible is proved to be γ.

V. Synthesis

There are several possible strategies for the synthesis of a Gain Scheduled LPV controller, and here three different approaches are presented, one in detail. There is no "better" or "worse" between the two, it all depends on the type of problem being treated, the system regulation requirements and computational tools available.

A. LFT Representation

This is a very useful model for LPV Systems, with structured and efficient algorithms for synthesis, where the complexity is comparable to the synthesis of a LTI controller [11]. However, one should be careful with this method, seen that it can be restrictive in several practical applications. An interesting recent approach for LFT systems is given by [15] where Parameter-Dependent Lyapunov functions are used for the synthesis.

B. LPV specific Synthesis Technique

LPV specific techniques can be found in several works, even though the approach is often different from one to another. The most appropriated from our point of view is the one in [12]. There are drawbacks as a possible dependence on the derivative of the parameter, which can be a problem for several applications.

The development of this technique is long and as this is not the goal of this present work, it will not be done here. But to make the synthesis of a LPV regulator, there are several *functionnal inequalities* to verify (in the general case, polytopic domains reduce these to a finite set), and assumptions must be made to avoid the dependence on the derivative.

The advantage of this method is obviously the fact that there are no iterations, and performance and stability specifications are considered from the start. The inconvenient is that is is more complex and computationally heavier than the *Ad Hoc* techniques.

C. *Ad Hoc Techniques*

The *Ad Hoc* techniques are those for which the performance and stability specifications are not considered *à priori*, but should be verified after the synthesis procedure. As it is intuitive to see, there is not an exact procedure for doing this, and it may vary with the experience of the engineer. From our point of view, this type of method shows the interest of Gain Scheduling: with very simple methods it is possible to make the synthesis of very good non-linear regulators.

Even if there are no models, it is possible to make some regulators with data tables issued from the plant behavior, but in this way there will not be analytical proofs of stability or performance. For this reason, in this section it is assumed in this section that the LPV Model was already obtained.

The difference between the procedure here described and the specific one resides mainly in a simpler synthesis phase. In addition, there is a greater choice of techniques for local regulator synthesis, where experience for particular applications can be used. Surprisingly, the procedure here described is not the "natural" procedure found in the literature, where usually one of the two cases arises: applications are justified with simulations or a specific LPV technique is used.

Once the LPV model is defined (1), a controller with the following structure is to be searched:

$$K(\delta(t)) : \begin{cases} \dot{x_K}(t) = A_K(\delta(t))x_K(t) + B_K(\delta(t))y(t) \\ u(t) = C_K(\delta(t))x_K(t) \end{cases} \tag{31}$$

So any non structured or non specific method that results in this representation, is an *Ad Hoc* technique. This choice is obviously vital for the global system, so, what is propose in this paper is the following technique. First, the synthesis phase:

- Depart from an open-loop LPV model
- Grid the scheduling variable domain for n local models (this should not be confused with griding a non-linear plant for the LPV modeling). The more points are used, the more the global performance will be close to the local evaluations, seen that the system will be closer to the synthesis points
- Perform the synthesis of n local regulators using a linear technique. The same topology shall be imposed to all the local regulators for the next step. For example, use a pole-placement technique for each chosen local model with the same number of poles/zeros.
- Proceed to the actual Gain Scheduling, the Scheduling law. As the same topology for each local model is imposed, a simple Gain/Pole/Zero interpolation results in an appropriated non-linear controller of the form (31). With this assumption there is no commutation, the function is continuous.
- Calculate the closed-loop.

This is a very light computational burden. But so far there is no proof of any kind. Simulations could be used for non-critical applications, but this is not the goal here. So, there is the analysis phase:

- Stability can be verified with a Parameter Dependent Lyapunov Function (Theorem (3.4)), seen that there is a closed-loop LPV model. In polytopic domains, the number of LMI is finite, and this is often the case. For example, as seen in [13], in a ship the domains of a two-dimensional Scheduling variable based on speed and angle between the ship and the current form a rectangle, which is a polytopic domain. This particular paper is an example of application without theoretical proof, where this particular procedure could had been used. Several other examples can be found for real applications, showing that this method of verification is suited.
- Performance is second to stability, given that stability is primordial. Three strategies are possible, and from our point of view the third one is the most appropriated:
  - Verify only local stability and simulate the global system. Global stability depends on local models, but this can be misleading, so for critical applications this is not the best choice, there can be an unexpected input that throws system's performance away from expected values.
  - Use the norms extensions given by (25) and (28). This is the ideal test, but can anyway be complicated to evaluate.
  - Use equation (30), in the first step, given that this test is based on a Parameter Dependent Lyapunov Function. With this test there is a minimum performance assurance. If the conditions are met for this test, it is possible to proceed to the simulation of the global system for the expected inputs, verifying if the outputs are reasonable.

If the analysis fails an iteration with another synthesis is necessary. Most of the time, global results will depend on the local models, so it is possible to know the direction to go during the local synthesis.

The obvious advantage of this method is the fact that with simple techniques, it is possible to build an analytically proved Gain Scheduled controller. An also obvious inconvenient is that it may take several iterations, and in the worst case scenario, one may never achieve the synthesis of the expected regulator. In this case, if possible, relaxing the system requirements may be a solution.

Anyway, as described in the preceding subsections, this technique is simpler than the specific one for LPV Systems. And in addition, it is better than what is currently used in industrial applications, where often simulations are the only proofs.

VI. Gain Scheduling Industrial Applications

Several applications of Gain Scheduling can be found through literature, in different domains. A robotic direct-drive manipulator is explored in [17], a ship control problem in [15], the rotating stall and surge control problem in [6] and an aerospace launch vehicle control problem in [5], to cite a few. From these few examples several domains are cited: robotics, naval and aerospatial industries and fluids mechanics.

It shows why it is important to find theoretical results for this approach of non-linear control.

VII. Conclusion and Future Works

In this work we were able to explore the different aspects of the Gain Scheduling methodology, from modeling to synthesis and analysis. Modeling is a crucial phase that we did not detail, but should not be neglected.

The procedure we proposed is something in between of the theoretical tools and simulations, and from our point of view it is useful. For our research in the control of Electrical Networks, which is a very particular type of system, we find this technique suitable.

Finally, we see that our efforts are justified throughout the large number of industrial applications of this concept.

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